First-order ordinary differential equations

Before we get involved trying to understand partial differential equations (PDEs), we'll review the highlights of the theory of ordinary differential equations (ODEs). We'll do this in such a way that we can begin to anticipate some of the methods we'll be using on PDEs later.

An ordinary differential equation gives a relationship between a function of one independent variable, say $y(x)$, its derivatives of various orders $y'(x)$, $y''(x)$ etc. and the independent variable x. The *order* of the equation is the order of the highest derivative that appears. So a first-order differential equation can always be put into the form:

$$F(x, y, y') = 0.$$

In general, it is possible to find solutions to such ODEs, and there is usually one and only one solution whose graph passes through a given point (a, b). In other words, there is one and only one solution of the *initial-value problem*:

$$F(x, y, y') = 0 \qquad y(a) = b.$$

At this level of generality, it's impossible to say much more. But there are several special types of first-order ODEs for which solution techniques are known (i.e., the separable, linear, homogeneous and exact equations from Math 240). We'll review the first two kinds, since we won't need the other two.

A first-order differential equation is called *separable* if it can be put in the form:

$$y' = f(x)g(y),$$

so that you can separate the variables as

$$\frac{dy}{g(y)} = f(x)\,dx$$

and then integrate both sides to get at least y as an implicitly-defined function of x. The constant of integration is then chosen so that the graph of the solution passes through the specified point in the initial-value problem. For instance, to solve

$$y' = xy \qquad y(0) = 1,$$

we separate:

$$\frac{dy}{y} = x\,dx$$

and integrate:

$$\int \frac{dy}{y} = \int x\,dx$$

to get

$$\ln y = \frac{x^2}{2} + C.$$

If $y = 1$ when $x = 0$, then we must have $C = 0$ and we can solve the resulting equation $\ln y = \frac{1}{2}x^2$ to get

$$y = e^{x^2/2}$$

as the solution of the initial-value problem.

A slight word of caution: there will be a technique in PDE called "separation of variables". It has *nothing* to do with the kind of separation for first-order ODEs.

Linearity is an important concept in many parts of mathematics. In the theory of differential equations (both ordinary and partial), we often think of the set of (differentiable) functions as comprising a vector space, since one can add two functions together to get another function, and one can multiply a function by a constant to get another function, just as one does with ordinary vectors. The addition and scalar multiplication of functions satisfies all the vector space axioms, so it is reasonable to think of functions as vectors.

And if functions are vectors, then what corresponds to matrices? In the case of ordinary vectors, we use matrices to represent linear transformations, and so we will consider operations on functions that have the basic linearity property

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$$

for all functions f and g and scalars (constants) α and β to be *linear operators*. Two basic examples of linear operators are

1. *Multiplication by a fixed function*: For instance, the operation "multiply by e^x" is a linear operation (we would write $L(f) = e^x f(x)$) because $e^x(\alpha f(x) + \beta(g(x))$ is clearly equal to $\alpha(e^x f(x)) + \beta(e^x g(x))$.
2. *Differentiation*: Differentiation is another linear operation, and when we're thinking of differentiation this way we'll denote it by D. And of course the sum and "constant-times-a-function" rules for derivatives imply that

$$D(\alpha f + \beta g) = \alpha D(f) + \beta D(g).$$

A more general linear operator would be a *first-order linear differential operator*. The most general such operator combines differentiation with multiplication by functions as follows:

$$L(f) = a(x)D(f) + b(x)f.$$

You should check that such an L is a linear operator.

A *linear first-order differential equation* has the form $L(u) = h(x)$, where $h(x)$ is a given function and we're trying to find the function $u(x)$. Notice the similary between this way of saying it and the linear algebra problem $Ax = b$ that you usually solve by Gaussian elimination. The standard way one solves a first-order linear ODE is as follows: First, divide both sides by $a(x)$, and set $p(x) = b(x)/a(x)$ and $q(x) = h(x)/a(x)$, so the resulting equation looks like

$$u' + p(x)u = q(x).$$

Then the trick is to "recognize" that the linear operator on the left, which is the sum of D and multiplication by $p(x)$, can also be written as the composition (product) of three operators, each of which is easy to invert:

$$u' + p(x)u = e^{-\int p}(D(e^{\int p}u)).$$

This is straightforward to check, and since multiplication by $e^{-\int p}$ is the inverse operation to multiplication by $e^{\int p}$, this last way of writing the linear operator is reminiscent of the similarity transformation of matrices $M^{-1}AM$ that is so useful in linear algebra. And once we write the equation as

$$e^{-\int p}(D(e^{\int p}u)) = q,$$

we can solve the equation by first multiplying both sides by $e^{\int p}$, then integrating both sides, and finally by multiplying both sides by $e^{-\int p}$. This yields the solution:

$$u = e^{-\int p}\left(\int e^{\int p} q\,dx + C\right) \tag{$*$}$$

and we can use the constant of integration C to satisfy any given initial condition.

As an example, let's solve

$$u' + \frac{1}{x}u = x^2 \qquad u(1) = 2.$$

According to the formula (*) above, the general solution of the equation is

$$u = e^{-\int 1/x}(\int e^{\int 1/x} x^2\,dx + C) = \frac{1}{x}(\frac{x^4}{4} + C).$$

In other words, $u = x^3/4 + C/x$. Since we're supposed to have $u(1) = 2$, we see that $C = 7/4$, and so the solution of the initial-value problem is

$$u = \frac{x^3}{4} + \frac{7}{4x}.$$

For the record, let's check that this works: we have

$$u' = \frac{3x^2}{4} - \frac{7}{4x^2}$$

and

$$\frac{1}{x}u = \frac{x^2}{4} + \frac{7}{4x^2},$$

so it's true that $u' + (1/x)u = x^2$ and of course the initial condition is also satisfied.

A question that arises is that of the *uniqueness* of the solution to this problem. Certainly $u = (x^3/4) + 7/(4x)$ is the only solution we get from formula (*), but how do we know there isn't some other solution of the initial-value problem that comes from a method we haven't considered? Uniqueness of solutions is an especially important issue when we *don't* have a formula or method for constructing a solution, since then we are forced to use some kind of approximation, and it is difficult to get an approximation to converge to something that is ambiguous or not uniquely defined (in other words, if there is more than one solution to the problem, how does the approximation method know which solution to approximate?). So we'll spend some time looking at uniqueness theorems for ODE problems, by way of anticipating the techniques we will be using for PDEs later.

So for initial-value problems for linear first-order ODEs, we have the following uniqueness theorem:

Theorem: There is one and only one solution to the initial-value problem

$$u' + p(x)u = q(x), \qquad u(a) = b$$

(on any interval containing a on which the functions $p(x)$ and $q(x)$ are defined and continuous).

Proof. The beginning of this uniqueness proof is paradigmatic for all uniqueness proofs for linear problems. We assume that there are two solutions $u_1(x)$ and $u_2(x)$ of the problem, and consider the difference $v(x) = u_1(x) - u_2(x)$. If we can show that $v(x) \equiv 0$ for all x, then u_1 will have to equal u_2, so there can be only one solution of the problem (since any two solutions will differ by zero).

To show that $v \equiv 0$, we note (as we will always note) that v satisfies the *related homogeneous problem*:

$$v' + p(x)v = 0, \qquad v(a) = 0.$$

This is easy to verify and uses the linearity of the differential operator on the left side of the equation in an essential way. Of course, the zero function $v(x) = 0$ for all x, is a solution to this homogeneous problem. We must show that there is no other solution. So let $v(x)$ be *any* solution to the homogeneous problem, and (inspired by the linear algebra considerations above) consider the function

$$w(x) = e^{\int p} v.$$

We have

$$w'(x) = e^{\int p} v' + e^{\int p} pv$$

by the product rule and the fact that the derivative of an integral is the integrand. But then $w' = e^{\int p}(v' + pv) = 0$, since v is a solution of the homogeneous problem. Thus $w' \equiv 0$ so that w must be a constant. What constant? Well, $w(a) = e^{\int p}(a)v(a)$, and $v(a) = 0$, so $w(a) = 0$ and thus $w \equiv 0$. And since e to any power is non-zero, we must therefore have $v \equiv 0$.

Now, unwinding the reasoning, since $v \equiv 0$, we have that $u_1 - u_2 \equiv 0$, or $u_1 = u_2$ for any pair of solutions of the original problem. In other words, there is *at most* one solution of the problem. But since we have the formula (*) for a solution, there is exactly one solution.

Second-order ordinary differential equations

The only part of the theory of second-order ODEs we will review is the part about homogeneous linear equations with constant coefficients. These are equations of the form

$$u'' + bu' + cu = 0,$$

where b and c are constants. Using our linear algebra notation from the previous section (where D is the derivative operator), we can write this as $Lu = 0$, where

$$Lu = D^2 + bD + cI$$

(I being the identity operator, so cI means simply to multiply the function by the constant c).

You probably recall that the solutions of this equation have something to do with e^{rx} for some constant(s) r that are determined by solving the quadratic equation $r^2 + br + c = 0$. But to *motivate* the impulse to look for a solution among functions of the form e^{rx}, it is useful to revert to linear-algebra-speak, in particular, to the lingo of eigenvalues and eigenvectors.

Recall that in linear algebra, we say that the (non-zero) vector $\mathbf{v}$ is an eigenvector of the matrix (linear operator) A corresponding to the eigenvalue λ if $A\mathbf{v} = \lambda\mathbf{v}$. Many linear algebra problems and theorems concerning the matrix A are simplified considerably if one works in a basis consisting (to the extent possible) of eigenvectors of A.

Another linear algebra concept to remember is the idea of the *kernel* of a linear transformation A – it is the set of vectors that A "maps to zero", in other words $\mathbf{v}$ is in the kernel of A if $A\mathbf{v} = \mathbf{0}$. Since sums and multiples of vectors in the kernel of A are also themselves in the kernel of A, the kernel of A is a vector space itself (so we can describe it by giving a basis for it). The concepts of eigenvalue and kernel come together in the statement that the kernel of A consists of the eigenvectors of A corresponding to the eigenvalue 0.

In the study of differential equations, we often think of functions as vectors as we did previously, and it is important to notice that the function e^{rx} is an eigenvector (eigenfunction?) of the linear operator D, corresponding to the eigenvalue r. In other words,

$$D(e^{rx}) = re^{rx}$$

for *every* constant r, real or complex (even $r = 0$, which gives the non-zero constants as eigenfunctions of D corresponding to the eigenvalue zero, or in other words, constants comprise the kernel of D). And when we are trying to solve the equation $u'' + bu' + cu = 0$, in other words, $Lu = 0$, we are looking for (a basis of) the kernel of the operator L.

We can think of L as being a polynomial in the operator D, in the sense that, since $Lu = D^2 + bD + cI$, we can think of L as $p(D)$, where $p(x) = x^2 + bx + c$. The advantage gained by this idea comes from linear algebra again: if A is a matrix and p is a polynomial, we can form $p(A)$ just as we formed $p(L)$ – if $p(x) = x^2 + bx + c$, then $p(A) = A^2 + bA + cI$. And if $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ, then $\mathbf{v}$ is also an eigenvector of $p(A)$ but corresponding to the eigenvalue $p(\lambda)$, because:

$$\begin{aligned}
p(A)\mathbf{v} &= (A^2 + bA + cI)\mathbf{v} = A^2\mathbf{v} + bA\mathbf{v} + cI\mathbf{v} \\
&= A(A\mathbf{v}) + b\lambda\mathbf{v} + c\mathbf{v} = A(\lambda\mathbf{v}) + b\lambda\mathbf{v} + c\mathbf{v} \\
&= \lambda A\mathbf{v} + b\lambda\mathbf{v} + c\mathbf{v} = \lambda(\lambda\mathbf{v}) + b\lambda\mathbf{v} + c\mathbf{v} \\
&= (\lambda^2 + b\lambda + c)\mathbf{v} = p(\lambda)\mathbf{v}
\end{aligned}$$

Applying this to the operator L, we see that $L(e^{rx}) = p(D)(e^{rx}) = p(r)e^{rx}$. So to find elements of the kernel of L, it's enough to find the values of r for which $p(r) = 0$. This is the method that is taught in Math 240 (although it is usually not particularly well motivated there).

Since p is a quadratic polynomial whose coefficients are real numbers, there are three possibilities:

1. The two roots of p are distinct real numbers r_1 and r_2, in which case the solution of $Lu = 0$ is $u = c_1e^{r_1x} + c_2e^{r_2x}$.

2. The two roots of p are complex, and are conjugates of each other - the real part of these roots will be $-b/2$ by the quadratic formula, and the imaginary part will be $\alpha = \sqrt{4c - b^2}$. Using Euler's formula concerning complex exponentials, we get that the solution of $Lu = 0$ is $u = e^{-bx/2}(c_1 \cos \alpha x + c_2 \sin \alpha x)$.

3. The two roots of p are equal (and equal to $-b/2$). In this case the solution of $Lu = 0$ is $u = c_1 e^{-bx/2} + c_2 x e^{-bx/2}$.

Now where did the function xe^{rx} come from in the third alternative above? One way to see that such a function must come up is to think of $L(e^{rx})$, which is equal to $p(r)e^{rx}$ as a function of the variable r. If we are in the third alternative, then $r = -b/2$ is a double root of $p(r)$, which means that not only is $p(-b/2) = 0$, but dp/dr evaluated at $r = -b/2$ must be zero as well. This means that the derivative of $p(r)e^{rx}$ with respect to r will be zero when $r = -b/2$. But since $p(r)e^{rx}$ also equals $L(e^{rx})$, the derivative with respect to r of $L(e^{rx})$ evaluated at $r = -b/2$ must be zero. Now, the only place there is an r in $L(e^{rx})$ is in the exponential – so

$$\frac{d}{dr}L(e^{rx}) = L(\frac{d}{dr}e^{rx}) = L(xe^{rx}).$$

Since we know that this is zero for $r = -b/2$ when we are in alternative three, we get that $xe^{-bx/2}$ is the "other" linearly independent solution of $Lu = 0$ in this case.

There are two arbitrary constants in our solution of $Lu = 0$ in any case. This makes sense, because we expect to have to integrate twice in order to solve a second-order differential equation. Thus, to formulate a problem having a unique solution, we expect to have to specify two pieces of data along with our second-order differential equation. One way to do this is to specify the value of u and of du/dx for a given value of x – this is the standard initial-value problem. We could also consider specifying "boundary values", i.e., values of u for two different values of x. Both types of problem are common in applications. As we did for first-order equations, let's prove uniqueness for these two types of problem (so we know that the solutions we've found by the above methods are the only ones.

First, let's consider the initial-value problem

$$u'' + cu = 0, \qquad u(0) = a, \quad u'(0) = b.$$

Assuming $c \geq 0$, we'll show that this has one and only one solution (it is also true if $c < 0$ but this must be proved by other means). We know how to find one solution – solve the polynomial $r^2 + c = 0$ and then use the arbitrary constants in the solution to match the values of a and b. So we have only to show that if there are two solutions u_1 and u_2 of this problem, they must in fact be the same.

As we did before, we begin by forming the difference $v(x) = u_1(x) - u_2(x)$. If we can show that $v(x) \equiv 0$, this will imply that $u_1 = u_2$. And it is easy to verify that v satisfies the "homogeneous" problem

$$v'' + cv = 0, \qquad v(0) = 0 \quad v'(0) = 0.$$

To show that v is identically zero, we'll consider the following "energy" function motivated by physical considerations:

$$E(x) = (v'(x))^2 + c(v(x))^2.$$

The first term of E should remind you of kinetic energy (proportional to velocity squared), and the second term is something like potential energy, and we're going to prove a "conservation of energy" principle in order to prove that $v \equiv 0$. To this end, we calculate:

$$\frac{dE}{dx} = 2v'(x)v''(x) + 2cv(x)v'(x) = 2v'(x)(v'' + cv),$$

which is identically zero, because v satisfies $v''+cv = 0$. Since its derivative is zero, we must have that E is constant, and therefore identically equal to zero, since $v(0) = 0$ and $v'(0) = 0$. But if $(v')^2 + cv^2 = 0$, then we must have $v' = v = 0$, since a each square is either positive or zero. Thus, v is identically zero.

The situation for boundary-value problems is more complicated. In fact, we do not always have uniqueness (or even existence) for boundary-value problems. For example, the solution of the differential equation $u'' + u = 0$ is $u = c_1 \cos x + c_2 \sin x$. If we specify $u(0) = 0$, then we must have $c_1 = 0$, and the function $c_2 \sin x$ vanishes at $x = \pi$ (or $x = 2\pi, 3\pi, \ldots$) no matter what c_2 is. Therefore the boundary-value problem

$$u'' + u = 0, \qquad u(0) = 0, \quad u(\pi) = b$$

has no solution if $b \neq 0$, and has infinitely many solutions (one for each value of c_2) if $b = 0$.

But we can prove uniqueness (which incidentally implies existence) of solutions of the following boundary-value problem:

$$u'' - cu = 0, \qquad u(a) = p, \quad u(b) = q$$

provided $c \geq 0$, for any values of a, b, p, q. We'll start the usual way, by assuming we have two solutions u_1 and u_2 and forming their difference $v = u_1 - u_2$. This function v will satisfy the homogeneous boundary value problem

$$v'' - cv = 0, \qquad v(a) = 0, \quad v(b) = 0.$$

We need to show that $v \equiv 0$ is the only solution of this homogeneous problem. But for any solution v, we can use integration by parts (with $f = v'(x)$ and $dg = v'(x)\,dx$) to calculate that

$$\begin{aligned}\int_a^b (v'(x))^2 dx &= v(x)v'(x)\big|_a^b - \int_a^b v(x)v''(x)\,dx \\ &= 0 - \int_a^b c(v(x))^2 dx \leq 0,\end{aligned}$$

since $v(a) = v(b) = 0$ and $v'' = cv$ by the differential equation. But the first integral on the left cannot be negative, so it must be zero. Thus v' is identically zero, so v is a constant and hence $v \equiv 0$ because $v(a) = v(b) = 0$. You can see that we used the assumption that $c \geq 0$ in an essential way. This trick of integrating by parts will come up many times as we study partial differential equations.